Further relations amongst infinite series and products. II. The evaluation of three-dimensional lattice sums



1990 J. Phys. A: Math. Gen. 23 117

(http://iopscience.iop.org/0305-4470/23/2/009)

J. Phys. A: Math. Gen. 23 (1990) 117-132. Printed in the UK

# Further relations amongst infinite series and products: II. The evaluation of three-dimensional lattice sums

I J Zucker

Department of Physics, King's College, London WC2R 2LS, UK

Received 22 March 1989

**Abstract.** New relations amongst infinite series and infinite products are given. A survey of these and previous results is presented. These relations provide a way of evaluating certain three-dimensional lattice sums in closed form. Over 50 such evaluations are displayed.

## 1. Introduction

In a previous communication (Zucker 1987, henceforth referred to as I) properties of the infinite series

$$\begin{aligned} &\theta(k,l)=\sum q^{(kn+l)^2} \qquad &&\bar{\theta}=\sum(-1)^n q^{(kn+l)^2} \\ &\phi(k,l)=\sum(-1)^{n(n-1)/2} q^{(kn+l)^2} \qquad &&\bar{\phi}=\sum(-1)^{n(n+1)/2} q^{(kn+l)^2} \end{aligned} \tag{1}$$

and their relations with the infinite products

$$Q(k,l)=\prod(1-q^{kn-l}) \qquad \bar{Q}(k,l)=\prod(1+q^{kn-l}) \tag{2}$$

were discussed. $\Sigma$ denotes summation over integer $n$ from $-\infty$ to $\infty$, whereas $\Pi$ denotes product over $n$ from one to $\infty$. The set of relations given in (1) will be referred to as $\Theta$. It was shown in I how to express $\Theta$ in terms of $Q$ and $\bar{Q}$. Here we shall augment $\Theta$ with another set of infinite series, $X$, defined by

$$\begin{aligned} &\chi(k,l)=\sum\cos\left(\frac{2\pi n}{3}\right)q^{(kn+l)^2} \qquad \bar{\chi}(k,l)=\sum(-1)^n\cos\left(\frac{2\pi n}{3}\right)q^{(kn+l)^2} \\ &\psi(k,l)=\sum(-1)^{n(n-1)/2}\cos\left(\frac{2\pi n}{3}\right)q^{(kn+l)^2} \\ &\bar{\psi}(k,l)=\sum(-1)^{n(n+1)/2}\cos\left(\frac{2\pi n}{3}\right)q^{(kn+l)^2} \end{aligned} \tag{3}$$

It will be shown here that certain members of $\Theta$ and $X$ may be expressed in a particularly apt form of infinite product. This, together with the fact that certain members of the set $\Theta'$ given by

$$\begin{aligned} &\theta'=\sum(kn+l)q^{(kn+l)^2} \qquad &&\bar{\theta}'=\sum(-1)^n(kn+l)q^{(kn+l)^2} \\ &\phi'=\sum(-1)^{n(n-1)/2}(kn+l)q^{(kn+l)^2} \qquad &&\bar{\phi}'=\sum(-1)^{n(n+1)/2}(kn+l)q^{(kn+l)^2} \end{aligned} \tag{4}$$

may also be expressed as infinite products, allows the evaluation of many three-dimensional lattice sums to be accomplished. It is not known whether the list given here is exhaustive, but the methods used here indicate clearly how other three- and many-dimensional sums might be evaluated in closed form.

Some further notational additions to those introduced in I have been made here. The particularly apt forms of infinite product in which $\Theta$ and $X$ may be expressed involve the Euler partition function

$$Q(k,0)=\prod(1-q^{kn})$$

and this will be written $(k)$. Sometimes a slightly modified form of this is required, namely the Dedekind $\eta$-function

$$\eta=q^{k/24}\prod(1-q^{kn})$$

and this will be written $[k]$. Jacobi was particularly fond of using the products

$$Q_0=\prod(1-q^{2n})\qquad Q_1=\prod(1+q^{2n})\qquad Q_2=\prod(1+q^{2n-1})\qquad Q_3=\prod(1-q^{2n-1}).$$

For conciseness these will be denoted $w$, $x$, $z$, $y$ respectively. A dictionary is easily established between the two sets of notation. Thus

$$w=(2)\qquad x=\frac{(4)}{(2)}\qquad y=\frac{(1)}{(2)}\qquad z=\frac{(2)^2}{(1)(4)}$$

from which the well known result $xyz=1$ is immediately obtained. Expressions for $\Theta$ and $X$ in terms of $(k)$ will be termed Eulerian, whereas if given in terms of $w$, $x$, $y$, $z$ they will be called Jacobian. When an expression has a Jacobian representation such that different parts of it have different powers of the variable $q$ as their argument, this will be denoted by a vertical bar with $q$ to the given power written after it, if it is not equal to $q$. Thus it will be shown later that

$$\theta(3,1)|q^{1/3}=q^{1/3}z|wxy|q^3. \tag{5}$$

This means that on the LHS of (5), $q$ has been replaced by $q^{1/3}$, whereas on the RHS, $q^{1/3}z$ is as it stands, but $wxy$ has argument $q^3$ for $q$.

In the course of deriving the results presented here, several simple operations were employed, and these are now described. $O(n)$ is the operation which replaces $q$ by $q^n$ in any expression. In particular, the following results were often used:

$$O(\tfrac{1}{2})w=wy\qquad O(\tfrac{1}{2})x=xz\qquad O(2)w=wx\qquad O(2)y=yz.$$

$S$ is the operation which changes $q$ to $-q$ in any expression. This leads to

$$Sw=w\qquad Sx=x\qquad Sy=z\qquad Sz=y$$

$$S(k)\begin{cases}(k) & \text{for even integer } k\\[4pt] \dfrac{(2k)^3}{(k)(4k)} & \text{for odd integer } k.\end{cases} \tag{6}$$

Expressions which are related by the $S$ operation will be termed sign-conjugate.

A further operation on $q$-series and products may be performed by means of the Poisson transformation denoted here by $P$. This transforms expressions in terms of $q=\mathrm{e}^{-\pi t}$ into those of a complementary variable $\rho=\mathrm{e}^{-\pi/t}$. The fundamental result for

$q$-series is

$$Pk\theta(k, l) = \frac{1}{\sqrt{t}} \sum \cos\left(\frac{2\pi n l}{k}\right) \rho^{n^2/k^2} \tag{7}$$

and, from relations amongst $\Theta$ given in I, the Poisson transform of any member of $\Theta$ may be found. They are as follows:

$$Pk\bar{\theta}(k, l) = \frac{1}{\sqrt{t}} \sum \cos\left((2n+1)\frac{\pi l}{k}\right) \rho^{(2n+1)^2/4k^2} \tag{8}$$

$$Pk\phi(k, l) = \frac{1}{\sqrt{t}} \sum \sin\left((2n+1)\frac{\pi}{4}\right) \cos\left[(2n+1)\pi\left(\frac{k+2l}{4k}\right)\right] \rho^{(2n+1)^2/16k^2} \tag{9}$$

$$Pk\bar{\phi}(k, l) = \frac{1}{\sqrt{t}} \sum \cos\left((2n+1)\frac{\pi}{4}\right) \sin\left[(2n+1)\pi\left(\frac{k+2l}{4k}\right)\right] \rho^{(2n+1)^2/16k^2} \tag{10}$$

For transforming products the most important result is expressed in terms of the Dedekind $\eta$ function. This is

$$P\eta[2k:q] = \frac{1}{\sqrt{kt}} \eta\left[\frac{2}{k}:\rho\right] \tag{11}$$

which enables one to transform any result in Eulerian form to another in that form, since the $P$ operation is multiplicatively transitive. That is, $P[a][b][c]\ldots = P[a]P[b]P[c]\ldots$. The Jacobian products transform thus

$$\begin{aligned} P[q^{1/12}w] &= \rho^{1/12}w \qquad & P[q^{1/12}x] &= 2^{-1/2}\rho^{-1/24}y \\ P[q^{-1/24}y] &= 2^{1/2}\rho^{1/12}x \qquad & P[q^{-1/24}z] &= \rho^{1/24}z. \end{aligned} \tag{12}$$

Expressions related by $P$ transformations will be termed Poisson conjugate. Since $q$ and $\rho$ are completely interchangeable, a Poisson transform of a $q$ relation will yield another $q$ relation unless the expression is self-conjugate.

One further transformation will be applied, but only to $q$-series. This is the Mellin transform defined by

$$M[f(t)] = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t)\, \mathrm{d}t. \tag{13}$$

Here the Mellin transform will be applied to all $q$-series after replacing $q$ by $\mathrm{e}^{-t}$. As examples of such transformations taken from $\Theta$ and $\Theta'$, consider first

$$\bar{\theta}(6, 1) = \sum (-1)^n q^{(6n+1)^2} = q - q^{5^2} - q^{7^2} + q^{11^2} \ldots.$$

Replacing $q$ by $\mathrm{e}^{-t}$ and taking the Mellin transform, we have

$$\begin{aligned} M\bar{\theta}(6, 1) &= \frac{1}{\Gamma(s)} \sum (-1)^n \int_0^\infty t^{s-1} \exp[-t(6n+1)^2]\, \mathrm{d}t \\ &= \sum \frac{(-1)^n}{(6n+1)^{2s}} = 1 - 5^{-2s} - 7^{-2s} + 11^{-2s} \ldots = L_{12}(2s) \end{aligned}$$

where $L_{12}$ is a positive parity Dirichlet $L$-series of degree $2s$. The properties of these series are described briefly in the appendix. Similarly, if we perform the same operations on $\bar{\theta}'(6,1)$, we have

$$\bar{\theta}'(6,1)=\sum(-1)^n(6n+1)q^{(6n+1)^2}=q+5q^{5^2}-7q^{7^2}-11q^{11^2}\ldots$$
$$M\bar{\theta}'(6,1)=1+5^{2s-1}-7^{2s-1}-11^{2s-1}\ldots=(1+3^{1-2s})L_{-4}(2s-1)$$

where $L_{-4}$ is a negative parity $L$-series of degree $2s-1$. In general, Mellin transforms of members of $\Theta$ and $X$ always yield positive parity $L$-series of degree $2s$, whereas Mellin transforms of members of $\Theta'$ and $X'$ give negative parity $L$-series of degree $2s-1$.

Some comments are appropriate here on the traditional notation generally accepted for the Jacobian $\theta$-functions as found in Whittaker and Watson (1946). These are

$$\theta_1(z,q)=2\sum_0^\infty \sin[(2n+1)z]q^{(n+1/2)^2}$$
$$\theta_2(z,q)=2\sum_0^\infty \cos[(2n+1)z]q^{(n+1/2)^2}$$
$$\theta_3=\sum\cos(2nz)q^{n^2}\qquad \theta_4=\sum(-1)^n\cos(2nz)q^{n^2}.$$

Most, and possibly all, of $\Theta$ and $X$ could be accommodated by this notation. However, here we wish to emphasise $q$ as a variable rather than as a parameter, with $z$ equal to zero. In this case the Jacobian series which play a prominent role are

$$\theta_1'=\sum_0^\infty(-1)^n(2n+1)q^{(n+1/2)^2}\qquad \theta_2=2\sum_0^\infty q^{(n+1/2)^2}$$
$$\theta_3=\sum q^{n^2}\qquad \theta_4=\sum(-1)^nq^{n^2}.$$

The expressions for $\theta_1'$ and $\theta_2$ lack the symmetry of the summation being carried out over all $n$. This may be remedied by writing

$$\theta_2=\sum q^{(2n+1)^2/4}=2\sum q^{(4n+1)^2/4}=\theta(2,1)|q^{1/4}=2\theta(4,1)|q^{1/4}$$
$$\theta_1'=2\sum(4n+1)q^{(4n+1)^2/4}=2\theta'(4,1)|q^{1/4}.$$

This led the author at one time (Glasser and Zucker 1980) to augment these functions with

$$\theta_5=2\sum(-1)^nq^{(4n+1)^2/4}=2\bar{\theta}(4,1)|q^{1/4}$$
$$\theta_6'=2\sum(-1)^n(4n+1)q^{(4n+1)^2}=2\bar{\theta}'(4,1)|q^{1/4}.$$

$\theta_6'$ was a particularly inapt choice of notation and will henceforth be discarded. $\theta_5$ at least demonstrated some symmetry amongst the traditional $\theta$-functions and will be retained. Thus the well known classical relation $2\theta_2\theta_3=\theta_2^2(q^{1/2})$ has its counterpart in $2\theta_2\theta_4=\theta_5^2(q^{1/2})$, and symmetries amongst $\theta_2$–$\theta_5$ are further displayed when $S$- and $P$-transformations are performed on them. Thus we have

$$S\theta_3=\theta_4\qquad S[\theta_2(q^{1/2})/q^{1/2}]=\theta_5(q^{1/2})/q^{1/2}$$
$$P\theta_3(q^n)=\frac{1}{\sqrt{nt}}\theta_3(\rho^{1/n})\qquad P\theta_5(q^n)=\frac{1}{\sqrt{2nt}}\theta_5(\rho^{1/4n})$$
$$P\theta_4(q^n)=\frac{1}{\sqrt{nt}}\theta_2(\rho^{1/n})\qquad P\theta_2(q^n)=\frac{1}{\sqrt{nt}}\theta_4(\rho^{1/n}).$$

All these transformations are readily carried out when $\theta_2$–$\theta_5$ are put in Eulerian or Jacobian form.

## 2. $\Theta$ and $X$ in Eulerian and Jacobian form

Table 1 displays members of $\Theta$ and $X$ which may be expressed in Eulerian form. In I it was shown how any member of $\Theta$ could be expressed in an infinite product form, but it is only for small values of $k$ and $l$ that these products are able to be made Eulerian. This is illustrated with the following two examples from table 1. Consider first $\theta(4, 1)$. From (15) in I it may be written

$$\theta(4,1) = q|Q(8,0)\bar{Q}(8,2)\bar{Q}(8,6)|q^4 = q|Q(4,0)\bar{Q}(4,1)\bar{Q}(4,3)|q^8$$
$$= q|Q(4,0)\bar{Q}(2,1)|q^8 = q|(4)\frac{(2)^2}{(1)(4)}|q^8 = q|wxz|q^8.$$

Hence $\theta(4,1)|q^{1/8} = q^{1/8}(2)^2/(1) = q^{1/8}wxz$.

As a second example $\bar{\theta}(3,1)$ will be examined. Again from (15) in I it is straightforward to write $\bar{\theta}(3,1) = q|Q(6,0)Q(6,1)Q(6,5)|q^3$. From (10) in I we have $Q(1,0) = Q(6,0)Q(6,1)Q(6,2)Q(6,3)Q(6,4)Q(6,5)$. hence

$$Q(6,1)Q(6,5) = \frac{Q(1,0)}{Q(6,0)Q(6,2)Q(6,3)Q(6,4)}.$$

But

$$Q(6,0)Q(6,2)Q(6,4) = Q(3,0)Q(3,1)Q(3,2)|q^2 = Q(1,0)|q^2 = (2)$$

and

$$Q(6,3) = Q(2,1)|q^3 = \left.\frac{Q(1,0)}{Q(2,0)}\right|q^3 = \frac{(3)}{(6)}.$$

Thus

$$\bar{\theta}(3,1) = q\left.\frac{(1)(6)^2}{(2)(3)}\right|q^3 \qquad \text{or} \qquad \bar{\theta}(3,1)|q^{1/3} = q^{1/3}\frac{(1)(6)^2}{(2)(3)} = q^{1/3}y|wxz|q^3.$$

In a similar fashion all the elements of $\Theta$ in table 1 were constructed. The application of the sign transformation to $\Theta$ always led to other elements in $\Theta$. However, on applying the Poisson transformation to $\Theta$, although elements of $\Theta$ were mainly obtained, in some cases elements of $X$ were found. An example of each case is given.

Consider first $\bar{\theta}(1,0) = (1)^2/(2) = [1]^2/[2]$. Taking Poisson transforms of both sides we obtain

$$P\bar{\theta}(1,0) = \frac{1}{\sqrt{t}}\sum \rho^{(2n+1)^2/4} \qquad \text{from (8)}$$

whereas

$$P\left[\frac{[1]^2}{[2]}\right] = 2\frac{[4]^2}{\sqrt{t}[2]} \qquad \text{from (11).}$$

Thus

$$\sum \rho^{(2n+1)^2/4} = 2\frac{[4]^2}{[2]} = 2\rho^{1/4}\frac{(4)^2}{(2)} = \theta(2,1)|\rho^{1/4}.$$

**Table 1.** Members of $\Theta$ and $X$ which may be expressed in Eulerian form.

| Series | Conventional | Eulerian | Jacobian | |
|---|---|---|---|---|
| $\theta(1,0)$ | $\theta_3$ | $\dfrac{(2)^5}{(1)^2(4)^2}$ | $wz^2$ | (T1.1) |
| $\bar\theta(1,0)$ | $\theta_4$ | $\dfrac{(1)^2}{(2)}$ | $wy^2$ | (T1.2$a$) |
| $\bar\theta(1,0)\|q^2$ | $\theta_4(q^2)$ | $\dfrac{(2)^2}{(4)}$ | $wyz$ | (T1.2$b$) |
| $\theta(2,1)\|q^{1/8}=2\theta(4,1)\|q^{1/8}$ | $\theta_2(q^{1/2})$ | $2q^{1/8}\dfrac{(2)^2}{(1)}$ | $2q^{1/8}wxz$ | (T1.3$a$) |
| $\theta(2,1)\|q^{1/4}=2\theta(4,1)\|q^{1/4}$ | $\theta_2$ | $2q^{1/4}\dfrac{(4)^2}{(2)}$ | $2q^{1/4}wx^2$ | (T1.3$b$) |
| $\bar\theta(4,1)\|q^{1/8}$ | $\frac{1}{2}\theta_5(q^{1/2})$ | $q^{1/8}\dfrac{(1)(4)}{(2)}$ | $q^{1/8}wxy$ | (T1.4) |
| $\theta(3,1)\|q^{1/3}$ | $\frac{1}{2}[\theta_3(q^{1/3})-\theta_3(q^3)]$ | $q^{1/3}\dfrac{(2)^2(3)(12)}{(1)(4)(6)}$ | $q^{1/3}z\|wxy\|q^3$ | (T1.5) |
| $\bar\theta(3,1)\|q^{1/3}$ | $\frac{1}{2}[\theta_4(q^3)-\theta_4(q^{1/3})]$ | $q^{1/3}\dfrac{(1)(6)^2}{(2)(3)}$ | $q^{1/3}y\|wxz\|q^3$ | (T1.6$a$) |
| $\bar\theta(3,1)\|q^{2/3}$ | $\frac{1}{2}[\theta_4(q^6)-\theta_4(q^{2/3})]$ | $q^{2/3}\dfrac{(2)(12)^2}{(4)(6)}$ | $q^{2/3}yz\|wx^2\|q^3$ | (T1.6$b$) |
| $\theta(6,1)\|q^{1/24}$ | $\frac{1}{2}[\theta_2(q^{1/6})-\theta_2(q^{3/2})]$ | $q^{1/24}\dfrac{(2)(3)^2}{(1)(6)}$ | $q^{1/24}xz\|wy^2\|q^3$ | (T1.7$a$) |
| $\theta(6,1)\|q^{1/12}$ | $\frac{1}{2}[\theta_2(q^{1/3})-\theta_2(q^3)]$ | $q^{1/12}\dfrac{(4)(6)^2}{(2)(12)}$ | $q^{1/12}x\|wyz\|q^3$ | (T1.7$b$) |
| $\phi(6,1)\|q^{1/24}$ | $\frac{1}{2}[\theta_5(q^{1/3})+\theta_5(q^3)]$ | $q^{1/24}\dfrac{(1)(4)(6)^5}{(2)^2(3)^2(12)^2}$ | $q^{1/24}xy\|wz^2\|q^3$ | (T1.8) |
| $\bar\theta(6,1)\|q^{1/24}$ | | $q^{1/24}(1)$ | $q^{1/24}wy$ | (T1.9$a$) |
| $\bar\theta(6,1)\|q^{1/12}$ | | $q^{1/12}(2)$ | $q^{1/12}w$ | (T1.9$b$) |
| $\bar\theta(6,1)\|q^{1/6}$ | | $q^{1/6}(4)$ | $q^{1/6}wx$ | (T1.9$c$) |
| $\bar\phi(6,1)\|q^{1/24}$ | | $q^{1/24}\dfrac{(2)^3}{(1)(4)}$ | $q^{1/24}wz$ | (T1.10) |
| $\chi(1,0)$ | $\frac{1}{2}[3\theta_3(q^9)-\theta_3]$ | $\dfrac{(1)(4)(6)^2}{(2)(3)(12)}$ | $wxy\|z\|q^3$ | (T1.11) |
| $\bar\chi(1,0)$ | $\frac{1}{2}[3\theta_4(q^9)-\theta_4]$ | $\dfrac{(2)^2(3)}{(1)(6)}$ | $wxz\|y\|q^3$ | (T1.12$a$) |
| $\bar\chi(1,0)\|q^2$ | $\frac{1}{2}[3\theta_4(q^{18})-\theta_4(q^2)]$ | $\dfrac{(4)^2(6)}{(2)(12)}$ | $wx^2\|yz\|q^3$ | (T1.12$b$) |
| $2\chi(2,1)\|q^{1/8}$ | $\frac{1}{2}[\theta_2(q^{1/2})-3\theta_2(q^{9/2})]$ | $q^{1/8}\dfrac{(1)^2(6)}{(2)(3)}$ | $q^{1/8}wy^2\|xz\|q^3$ | (T1.13$a$) |
| $2\chi(2,1)\|q^{1/4}$ | $\frac{1}{2}[\theta_2-3\theta_2(q^9)]$ | $q^{1/4}\dfrac{(2)^2(12)}{(4)(6)}$ | $q^{1/4}wyz\|x\|q^3$ | (T1.13$b$) |
| $2\bar\psi(2,1)\|q^{1/8}$ | $\frac{1}{2}[\theta_5(q^{1/2})+3\theta_5(q^{9/2})]$ | $q^{1/8}\dfrac{(2)^5(3)(12)}{(1)^2(4)^2(6)^2}$ | $q^{1/8}wz^2\|xy\|q^3$ | (T1.14) |

This in conventional notation is the well known result $P\theta_4 = \theta_2$. When the same operations are applied to $\theta(3,1)$, however, we have

$$\theta(3,1) = q\frac{(6)^2(9)(36)}{(3)(12)(18)} = \frac{[6]^2[9][36]}{[3][12][18]}.$$

Thus

$$P\theta(3,1) = \frac{1}{3\sqrt{t}}\sum \cos\left(\frac{2\pi n}{3}\right)\rho^{n^2/9} = \frac{1}{3\sqrt{t}}\chi(1,0)|\rho^{1/9} \qquad \text{by (7)}$$

$$P\frac{[6]^2[9][36]}{[3][12][18]} = \frac{1}{\sqrt{t}}\,\frac{(1/3)(\sqrt{2/9})(\sqrt{1/18})}{(\sqrt{2/3})(\sqrt{1/6})(\sqrt{1/9})}\,\frac{[2/3]^2[4/9][1/9]}{[4/3][1/3][2/9]} \qquad \text{using (11).}$$

Equating these two expressions and replacing $\rho^{1/9}$ by $q$, we obtain

$$\chi(1,0) = \frac{(1)(4)(6)^2}{(2)(3)(12)}.$$

Following the procedures just described, all the members of sets $\theta$ and $X$ given in table 1 were constructed. In table 2 the sign, Poisson and Mellin transforms of these series are given. As previously described, the Mellin transforms of all the series in table 1 yield a Dirichlet $L$-series whose character depends on the series. Since $L$-series

**Table 2.** The sign, Poisson and Mellin transforms of the series $\Theta$ and $X$. The argument of the $L$-series is $(2s)$.

| Series | Sign Transform | Poisson Transform | Mellin Transform | |
|---|---|---|---|---|
| $\theta(1,0)$ | $\bar\theta(1,0)$ | $\theta(1.0)$ | $2L_1$† | (T2.1) |
| $\bar\theta(1,0)$ | $\theta(1,0)$ | $\theta(2,1)\|q^{1/4}$ | $-2(1-2^{1-2s})L_1$† | (T2.2) |
| $\theta(2,1)\|q^{1/8}$ | $2\bar\theta(4,1)\|q^{1/8}$ | $2^{1/2}\bar\theta(1,0)\|q^2$ | $2^{1+3s}(1-2^{-2s})L_1$ | (T2.3) |
| $\theta(3,1)\|q^{1/3}$ | $\bar\theta(3,1)\|q^{1/3}$ | $3^{-1/2}\chi(1,0)\|q^{1/3}$ | $3^s(1-3^{-2s})L_1$ | (T2.4) |
| $\bar\theta(3,1)\|q^{1/3}$ | $\theta(3,1)\|q^{1/3}$ | $2\times 3^{-1/2}\chi(2,1)\|q^{1/12}$ | $3^s(1-2^{1-2s})(1-3^{-2s})L_1$ | (T2.5) |
| $\bar\theta(4,1)\|q^{1/8}$ | $\theta(4,1)\|q^{1/8}$ | $\bar\theta(4,1)\|q^{1/8}$ | $8^sL_8$ | (T2.6) |
| $\theta(6,1)\|q^{1/24}$ | $\phi(6,1)\|q^{1/24}$ | $6^{1/2}\times 3^{-1}\bar\chi(1,0)\|q^{2/3}$ | $24^s(1-2^{-2s})(1-3^{-2s})L_1$ | (T2.7) |
| $\bar\theta(6,1)\|q^{1/24}$ | $\bar\phi(6,1)\|q^{1/24}$ | $2^{1/2}\bar\theta(6,1)\|q^{1/6}$ | $24^sL_{12}$ | (T2.8) |
| $\phi(6,1)\|q^{1/24}$ | $\theta(6,1)\|q^{1/24}$ | $3^{-1/2}\bar\psi(2,1)\|q^{1/24}$ | $24^s(1+3^{-2s})L_8$ | (T2.9) |
| $\bar\phi(6,1)\|q^{1/24}$ | $\bar\theta(6,1)\|q^{1/24}$ | $\bar\phi(6,1)\|q^{1/24}$ | $24^sL_{24}$ | (T2.10) |
| $\chi(1,0)$ | $\bar\chi(1,0)$ | $\theta(3,1)\|q^{1/9}$ | $-(1-3^{1-2s})L_1$ | (T2.11) |
| $\bar\chi(1,0)$ | $\chi(1,0)$ | $\theta(6,1)\|q^{1/36}$ | $(1-3^{1-2s})(1-2^{1-2s})L_1$ | (T2.12) |
| $2\chi(2,1)\|q^{1/8}$ | $2\bar\psi(2,1)\|q^{1/8}$ | $\bar\theta(3,1)\|q^{2/9}$ | $8^s(1-3^{1-2s})(1-2^{-2s})L_1$ | (T2.13) |
| $2\bar\psi(2,1)\|q^{1/8}$ | $2\chi(2,1)\|q^{1/8}$ | $\phi(6,1)\|q^{1/72}$ | $8^s(1+3^{1-2s})L_8$ | (T2.14) |

† These are actually the Mellin transforms of $\theta(1,0)-1$ and $\bar\theta(1,0)-1$.

of different character are algebraically independent, the independence or otherwise of the various elements of $\Theta$ and $X$ may easily be established.

In table 3 certain products of the elements of $\Theta$ and $X$ which exhibit particularly simple Eulerian and Jacobian structures are given. Since the history of expressing infinite series in product form and vice versa goes back to Euler, it has not been possible to ascertain with certainty the original sources of the results given in table 1. After a considerable literature search, it is believed that the following attributions of the results in table 1 are correct. Result (T1.1) is due to Jacobi. Results (T1.2) and (T1.3) are due to Gauss. Result (T1.9) is Euler's classic result, the very first of its kind. Results (T1.6) and (T1.7) have been given by Kac (1978), who claims them as new. He also gives (T1.11) and (T1.13) as new, but these results go back to Ramanujan (1957). Results (T1.4) and (T1.10) as given here are new, though (T1.4) is implied by Zucker (1975). However, Smith (1865) quotes some results of Jacobi from which (T1.4) and (T1.10) may be deduced. As far as we know (T1.5), (T1.8), T1.12) and (T1.14) are new as given here, though Borwein and Borwein (1986) establish something equivalent to (T1.12) in terms of conventional notation.

The interplay between the results in table 1 is often bizarre. Thus, at first sight the relation between $\phi(6, 1)$ and $\bar{\theta}(4, 1)$ implied by their Mellin transforms containing the same $L$-series is not at all apparent. But the relationship may be established as follows.

**Table 3.** Certain products of elements of $\Theta$ and $X$ which exhibit particularly simple Eulerian and Jacobian structures.

| Combination | Eulerian form | Jacobian form | |
|---|---|---|---|
| $\bar{\theta}(3,1)\|q^{1/9}\theta(6,1)\|q^{1/72}=2\bar{\chi}(1,0)\|\chi(2,1)\|q^{1/8}$ | $q^{1/8}(1)(2)$ | $q^{1/8}w^2y$ | (T3.1) |
| $\theta(3,1)\|q^{1/9}\phi(6,1)\|q^{1/72}=2\chi(1,0)\|\bar{\psi}(2,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(2)^4}{(1)(4)}$ | $q^{1/8}w^2z$ | (T3.2) |
| $\theta(8,1)\theta(8,3)\|q^{1/16}=\theta(4,1)\|q^{1/8}\theta(4,1)\|q^{1/2}$ | $q^{5/8}\dfrac{(2)^2(8)^2}{(1)(4)}$ | $q^{5/8}z\|w^2x^2\|q^2$ | (T3.3) |
| $\bar{\theta}(8,1)\bar{\theta}(8,3)\|q^{1/16}=\bar{\theta}(4,1)\|q^{1/8}\theta(4,1)\|q^{1/2}$ | $q^{5/8}\dfrac{(1)(8)^2}{(2)}$ | $q^{5/8}y\|w^2x^2\|q^2$ | (T3.4) |
| $\phi(4,1)\bar{\phi}(4,1)\|q^{1/16}=\theta(1,0)\|q^2\bar{\theta}(4,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(1)(4)^6}{(2)^3(8)^2}$ | $q^{1/8}y\|w^2z^2\|q^2$ | (T3.5) |
| $\theta(1,0)\|q^2\theta(4,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(4)^5}{(1)(8)^2}$ | $q^{1/8}z\|w^2z^2\|q^2$ | (T3.6) |
| $\theta(3,1)\bar{\theta}(3,1)\|q^{1/6}=\bar{\theta}(1,0)\|q^3\bar{\theta}(3,1)\|q^{1/3}$ | $q^{1/3}\dfrac{(1)(3)(6)}{(2)}$ | $q^{1/3}y\|w^2y\|q^3$ | (T3.7) |
| $\theta(1,0)\|q^3(3,1)\|q^{1/3}$ | $q^{1/3}\dfrac{(2)^2(6)^4}{(1)(4)(3)(12)}$ | $q^{1/3}z\|w^2z\|q^3$ | (T3.8) |
| $\theta(6,1)\phi(6,1)\|q^{1/48}=\bar{\theta}(1,0)\|q^3\theta(6,1)\|q^{1/24}$ | $q^{1/24}\dfrac{(2)(3)^4}{(1)(6)^2}$ | $q^{1/24}xz\|w^2y^4\|q^3$ | (T3.9) |
| $\theta(1,0)\|q^3\phi(6,1)\|q^{1/24}$ | $q^{1/24}\dfrac{(1)(4)(6)^{10}}{(2)^2(3)^4(12)^4}$ | $q^{1/24}xy\|w^2z^4\|q^3$ | (T3.10) |

From table 1 we have

$$\theta(3,1)+\chi(1,0)=\theta_3(q^9)=q\frac{(6)^2(9)(36)}{(3)(12)(18)}+\frac{(1)(4)(6)^2}{(2)(3)(12)}=\frac{(18)^5}{(9)^2(18)^2}.$$

Multiplying both sides by $q^{1/8}(3)(12)/(6)^2$ gives

$$q^{1/8}\frac{(3)(12)(18)^5}{(6)^2(9)^2(36)^2}=q^{9/8}\frac{(9)(36)}{(18)}+q^{1/8}\frac{(1)(4)}{(2)}$$

or

$$\phi(6,1)|q^{1/8}=\tfrac{1}{2}[\bar{\theta}(4,1)|q^{9/8}+\bar{\theta}(4,1)|q^{1/8}] \tag{14}$$

which is equivalent to (T1.8). Other similar connections may be found.

## 3. $\Theta'$ and $X'$ in Eulerian and Jacobian form

It has not been found possible to express $\Theta'$ and $X'$ in infinite product form as was done for $\Theta$. However, $\Theta'$ may be found in terms of Lambert series, and if these are recognised, then sometimes these may be expressed as infinite products. This is illustrated now for $\bar{\theta}'(k,l)$. The Jacobi triple product identify may be written

$$\sum(-1)^n a^n q^{n^2}=(2)\prod(1-aq^{2n-1})(1-a^{-1}q^{2n-1}). \tag{15}$$

The following operations are carried out in this order: (i) replace $a$ by $a^k$; (ii) multiply both sides of (15) by $a^l$; (iii) differentiate logarithmically with respect to $a$, and multiply by $a$; (iv) replace $q$ by $q^{k^2}$ and $a$ by $q^{2l}$. Then it is found that

$$\frac{\bar{\theta}'(k,l)}{\bar{\theta}(k,l)}=1+k\left(\sum_1^{\infty}\frac{q^{2nk^2-k^2-2kl}}{(1-q^{2nk^2-k^2-2kl})}-\frac{q^{2nk^2-k^2+2kl}}{(1-q^{2nk^2-k^2+2kl})}\right). \tag{16}$$

Similar expressions for other members of $\Theta'$ may be found. Series such as those found on the RHS of (16) are known as Lambert series. In order to identify them Mellin transforms are formed. For example, consider $k=4$, and $l=1$. We have

$$\frac{\bar{\theta}'(4,1)}{\bar{\theta}(4,1)}|q^{1/8}-1=4\left(\sum_1^{\infty}\frac{q^{4n-3}}{1-q^{4n-3}}-\frac{q^{4n-1}}{1-q^{4n-1}}\right)$$

$$=4\sum_1^{\infty}\sum_1^{\infty}(q^{(4n-3)m}-q^{(4n-1)m}). \tag{17}$$

The Mellin transform of the RHS of (17) is $4L_1(s)L_{-4}(s)$. But it is a well known result (Hardy and Wright 1980) going back to Lorenz (1871) and implicit in Jacobi that

$$M[\theta_3^2-1]=M[\theta^2(1,0)-1]=4L_1(s)L_{-4}(s).$$

So the RHS of (17) is just $\theta^2(1,0)-1$, and thus

$$\bar{\theta}'(4,1)|q^{1/8}=\bar{\theta}(4,1)|q^{1/8}\theta^2(1,0).$$

Then, using the results of table 1, $\bar{\theta}'(4,1)$ may be put into Eulerian and Jacobian forms. As a further example, consider $k=6$, and $l=1$. This gives

$$\left.\frac{\bar{\theta}'(6,1)}{\bar{\theta}(6,1)}\right|q^{1/24}-1=6\left(\sum_1^{\infty}\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right). \tag{18}$$

The Mellin transform of the RHS of (18) yields $6L_1(s)L_{-3}(s)$, which is also $M[\theta_3\theta_3(q^3)+\theta_2\theta_2(q^3)-1]$ (Zucker and Robertson 1976). Hence

$$\bar{\theta}'(6,1)|q^{1/24}=\bar{\theta}(6,1)|q^{1/24}[\theta_3\theta_3(q^3)+\theta_2\theta_2(q^3)].$$

This is then the sum of two Eulerian products, and it does not seem possible to collapse these into a single form.

Following the procedures just described, and also using sign and Poisson transformations, the results collated in table 4 were derived. Again, relations among some of these expressions which are apparent from their Mellin transforms are not immediately evident from their Eulerian forms. Thus the Mellin transforms of $\theta'(3,1)$, $\bar{\theta}'(3,1)$ and

**Table 4.** Transforms of various series. The argument of the $L$-series is $(2s-1)$.

| Series | Eulerian form | Jacobian form | Mellin transform | |
|---|---|---|---|---|
| $\theta'(3,1)\|q^{1/3}$ | $q^{1/3}\dfrac{(1)^2(4)^2}{(2)}$ | $q^{1/3}w^3x^2y^2$ | $3^sL_{-3}$ | (T4.1) |
| $\bar{\theta}'(3,1)\|q^{1/3}$ | $q^{1/3}\dfrac{(2)^5}{(1)^2}$ | $q^{1/3}w^3x^2z^2$ | $3^s(1+2^{2-2s})L_{-3}$ | (T4.2) |
| $\theta'(4,1)\|q^{1/8}$ | $q^{1/8}(1)^3$ | $q^{1/8}w^3y^3$ | $2^{3s}L_{-4}$ | (T4.3) |
| $\bar{\theta}'(4,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(2)^9}{(1)^3(4)^3}$ | $q^{1/8}w^3z^3$ | $2^{3s}L_{-8}$ | (T4.4) |
| $\theta'(6,1)\|q^{1/24}$ | $q^{1/24}\dfrac{(1)^5}{(2)^2}$ | $q^{1/24}w^3y^5$ | $24^s(1+2^{1-2s})L_{-3}$ | (T4.5) |
| $\phi'(6,1)\|q^{1/24}$ | $q^{1/24}\dfrac{(2)^{13}}{(1)^5(4)^5}$ | $q^{1/24}w^3z^5$ | $24^sL_{-24}$ | (T4.5) |
| $\bar{\theta}'(6,1)\|q^{1/8}$ | $q^{1/8}(1)^3+3q^{9/8}(9)^3$ | | $2^{3s}(1+3^{1-2s})L_{-4}$ | (T4.7) |
| $\bar{\phi}'(6,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(2)^9}{(1)^3(4)^3}-3q^{9/8}\dfrac{(18)^9}{(9)^3(36)^3}$ | | $2^{3s}(1-3^{1-2s})L_{-8}$ | (T4.8) |
| $2\bar{\chi}'(2,1)q^{1/8}$ | $q^{1/8}(1)^3+9q^{9/8}(9)^3$ | | $2^{3s}(1+3^{2-2s})L_{-4}$ | (T4.9) |
| $2\psi'(2,1)\|q^{1/8}$ | $q^{1/8}\dfrac{(2)^9}{(1)^3(4)^3}-9q^{9/8}\dfrac{(18)^9}{(9)^3(36)^3}$ | | $2^{3s}(1-3^{2-2s})L_{-8}$ | (T4.10) |

$\theta'(6,1)$ all contain $L_{-3}$, and hence must be related. This may be demonstrated from two classical results for $\theta$ functions. The first is $\theta_3=\theta_3(q^4)+\theta_2(q^4)$ which, if written in Eulerian form, is

$$\frac{(2)^5}{(1)^2(4)^2}=\frac{(8)^5}{(4)^2(16)^2}+2q\frac{(16)^2}{(8)}.$$

Multiplying both sides by $q^{1/3}(4)^2$ gives

$$q^{1/3}\frac{(2)^5}{(1)^2}=a^{1/3}\frac{(8)^5}{(16)^2}+2q^{4/3}\frac{(4)^2(16)^2}{(8)}$$

or

$$\bar{\theta}'(3,1)|q^{1/3}=\theta'(6,1)|q^{1/3}+2\theta'(3,1)|q^{4/3}.$$

Similarly, from the classical result $2\theta_3(q^4)=\theta_3+\theta_4$, we obtain

$$2\theta'(6,1)|q^{1/3}=\theta'(3,1)|q^{1/3}+\bar{\theta}'(3,1)|q^{1/3}.$$

The results in table 4 also have a long history. The oldest is Jacobi's famous result (T4.3) which is given in Hardy and Wright (1980) as

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}=\prod(1-q^n)^3=(1)^3.$$

As indicated previously, this may be put into the more symmetric form

$$\theta'(4,1)=\sum(4n+1)q^{(4n+1)^2}=q(8)^3=[8]^3$$

as it appears in table 4. Results (T4.1) and (T4.5) are usually ascribed to Gordon (1961). However, they go back at least to Ramanujan (1916), who quotes them along with Jacobi's result without reference, as though they were well known. But formulae given by Ramanujan without comment imply several possibilities. It may be (a) he did not know a reference, (b) he thought the formulae too well known to need a reference or (c) that the formulae were his own, or any mixture of the above. Thus (T4.1) and (T4.5) may well have originated with Ramanujan, and up to now no earlier source has been found. Result (T4.2) is given by Kac (1978) and is simply a sign transform of (T4.1). Result (T4.4) or its equivalent first seems to have appeared in Glasser and Zucker (1980), and is simply a sign transform of (T4.3). Result (T4.6) as presented here is new. It was suggested by Borwein and Borwein (1986) that performing the equivalent of a sign transformation on what was essentially $\theta'(6,1)$ would yield an interesting relation. The result is in fact $\phi'(6.1)$ and (T4.6). The other results in table 4 are new, and have been obtained by inverting the Mellin transforms of the series involved.

## 4. The evaluation of three-dimensional sums

It will be observed from tables 1 and 4 that $\Theta$ and $X$ contain $w$ to a single power whereas $\Theta'$ and $X'$ contain $w^3$. It is thus possible to combine three suitable members of $\Theta$ and $X$ together to form a $\Theta'$ or $X'$. For example, remembering that $xyz=1$,

$$\theta(1,0)\bar{\theta}(1,0)|\theta(4,1)|q^{1/4}=q^{1/4}wz^2wy^2wx^2=q^{1/4}w^3=\theta'(4,1)|q^{1/4}.$$

In classical notation this is Jacobi's expression

$$\theta_2\theta_3\theta_4=\theta_1'.$$

Written out fully this is

$$\sum\sum\sum(-1)^m q^{m^2+n^2+(4p+1)^2/4}=\sum(4n+1)q^{(4n+1)^2/4}.$$

Now replacing $q$ by $e^{-t}$ and taking Mellin transforms of both sides, we obtain

$$\sum\sum\sum(-1)^m[m^2+n^2+(4p+1)^2/4]^{-s}=4^sL_{-4}(2s-1). \tag{19}$$

Thus the three-dimensional sum on the LHS of (19) has been expressed in closed form, as was pointed out by Glasser (1973). Another example is

$$[\bar{\theta}(6,1)]^3=\theta'(4,1)|q^3$$

or

$$\sum\sum\sum(-1)^{m+n+p}[(6m+1)^2+(6n+1)^2+(6p+1)^2]^{-s}=3^{-s}L_{-4}(2s-1). \qquad (20)$$

This result was given by Forrester and Glasser (1982). It is clear then that any combination of results in tables 1 and 2 which yield a result in table 4 will lead to the evaluation of a three-dimensional sum in terms of a single Dirichlet $L$-series. In table 5 a list of some 60 such evaluations has been given. It is not known whether this list is exhaustive or not, or whether any given result is a trivial consequence of another. For example the fact that

$$\sum q^{(2n+1)^2}=2\sum q^{(4n+1)^2}$$

would enable many of the entries in table 5 to be augmented, but these have not been included. Several results in table 5 not immediately obvious from the entries in tables 1 and 2 have been obtained as sign or Poisson transforms of other results. For integer $s$ all these three-dimensional sums can be expressed exactly in terms of powers of $\pi$ and surds, since $L_{-d}(2s-1)=R\pi^{2s-1}\sqrt{d}$ where $R$ is a rational number (see the appendix).

**Table 5.** Three-dimensional sums evaluated in terms of single Dirichlet $L$-series. In this table $\sum$ implies summation over all the indices $m$, $n$, $p$ from $-\infty$ to $\infty$. We use the notation $A=L_{-4}(2s-1)$, $B=L_{-3}(2s-1)$, $C=(1+2^{2-2s})L_{-3}(2s-1)$, $D=(1+2^{1-2s})L_{-3}(2s-1)$, $E=L_{-8}(2s-1)$.

$$\sum(-1)^m[4m^2+4n^2+(4p+1)^2]^{-s}=A$$

$$\sum(-1)^{m+n}[8m^2+8n^2+(4p+1)^2]^{-s}=A$$

$$\sum(-1)^{m+n+p}[8m^2+16n^2+(4p+1)^2]^{-s}=A$$

$$\sum(-1)^m[8m^2+(4n+1)^2+(4p+1)^2]^{-s}=2^{-s}A$$

$$\sum(-1)^{m+n}[(4m+1)^2+(4n+1)^2+8p^2]^{-s}=2^{-s}A$$

$$\sum(-1)^{m+n}[16m^2+(4n+1)^2+(4p+1)^2]^{-s}=2^{-s}A$$

$$\sum(-1)^{m+n+p}[(6m+1)^2+(6n+1)^2+(6p+1)^2]^{-s}=3^{-s}A$$

$$\sum(-1)^{m+n+p}[24m^2+(6n+1)^2+2(6p+1)^2]^{-s}=3^{-s}A$$

$$\sum(-1)^m[(4m+1)^2+(4n+1)^2+2(4p+1)^2]^{-s}=4^{-s}A$$

$$\sum(-1)^{m+n}[(6m+1)^2+2(6n+1)^2+3(4p+1)^2]^{-s}=6^{-s}A$$

$$\sum(-1)^{m+n+[p(p+1)/2]}[3(4m+1)^2+2(6n+1)^2+(6p+1)^2]^{-s}=6^{-s}A$$

$$\sum(-1)^{m+n+[p(p+1)/2]}[(6m+1)^2+4(6n+1)^2+(6p+1)^2]^{-s}=6^{-s}A$$

$$\sum(-1)^m[4(3m+1)^2+4(3n+1)^2+(6p+1)^2]^{-s}=9^{-s}A$$

$$\sum(-1)^{m+n}[72m^2+8(3n+1)^2+(6p+1)^2]^{-s}=9^{-s}A$$

$$\sum(-1)^m[8(3m+1)^2+9(4n+1)^2+(6p+1)^2]^{-s}=18^{-s}A$$

$$\sum(-1)^{m+[n(n-1)/2]}[16(3m+1)^2+(6n+1)^2+(6p+1)^2]^{-s}=18^{-s}A$$

$$\sum(-1)^{m+[n(n-1)/2]}[9(4m+1)^2+(6n+1)^2+8(3p+1)^2]^{-s}=18^{-s}A$$

$$\sum(-1)^m\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)\cos\left(\frac{2\pi p}{3}\right)[4m^2+4n^2+(2p+1)^2]^{-s}=2^{-1}A$$

**Table 5.** (continued)

$$\sum(-1)^{m+n}\cos\left(\frac{2\pi n}{3}\right)\cos\left(\frac{2\pi p}{3}\right)[8m^2+8n^2+(2p+1)^2]^{-s}=2^{-1}A$$

$$\sum(-1)^{m}\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)[8m^2+(2n+1)^2+(4p+1)^2]^{-s}=2^{-1-s}A$$

$$\sum(-1)^{m+[n(n+1)/2]}\cos\left(\frac{2\pi n}{3}\right)\cos\left(\frac{2\pi p}{3}\right)[(4m+1)^2+(2n+1)^2+8p^2]^{-s}=2^{-1-s}A$$

$$\sum(-1)^{m+[n(n+1)/2]}\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)\cos\left(\frac{2\pi p}{3}\right)[16m^2+(2n+1)^2+(2p+1)^2]^{-s}$$
$$=2^{-2-s}A$$

$$\sum[2-(-1)^{m+n}](-1)^p[24m^2+72n^2+(6p+1)^2]^{-s}=(1+3^{1-2s})A$$

$$\sum[1+(-1)^{m+n}](-1)^p[2m^2+6n^2+(6p+1)^2]^{-s}=2(1+3^{2-2s})A$$

$$\sum[2-(-1)^{m+n}](-1)^p[8m^2+24n^2+(6p+1)^2]^{-s}=(1+3^{2-2s})A$$

$$\sum[1+(-1)^{m+n}](-1)^p[6m^2+18n^2+(6p+1)^2]^{-s}=2(1+3^{1-2s})A$$

$$\sum(-1)^{m+n+p}[3m^2+12n^2+(6p+1)^2]^{-s}=B$$

$$\sum(-1)^{m+n+p}[6m^2+(6n+1)^2+(6p+1)^2]^{-s}=2^{-s}B$$

$$\sum(-1)^{m+n}[12m^2+(6n+1)^2+3(4p+1)^2]^{-s}=4^{-s}B$$

$$\sum(-1)^{m+n+[p(p+1)/2]}[12m^2+3(4n+1)^2+(6p+1)^2]^{-s}=4^{-s}B$$

$$\sum(-1)^{m+n}[(6m+1)^2+(6n+1)^2+6(4p+1)^2]^{-s}=8^{-s}B$$

$$\sum(-1)^{m+n+p}[3(4m+1)^2+(4n+1)^2+2(6p+1)^2]^{-s}=8^{-s}B$$

$$\sum(-1)^{m+n+p}[3(4m+1)^2+(6n+1)^2+4(6p+1)^2]^{-s}=8^{-s}B$$

$$\sum(-1)^{m}[(6m+1)^2+3(4n+1)^2+12(4p+1)^2]^{-s}=16^{-s}B$$

$$\sum(-1)^{m+[n(n+1)/2]}[3(4m+1)^2+(6n+1)^2+12(4p+1)^2]^{-s}=16^{-s}B$$

$$\sum(-1)^{m}[2(6m+1)^2+3(8n+1)^2+3(8p+3)^2]^{-s}=32^{-s}B$$

$$\sum(-1)^{m+n+[p(p+1)/2]}[3(8m+1)^2+3(8n+3)^2+2(6p+1)^2]^{-s}=32^{-s}B$$

$$\sum(-1)^{m+n}[12m^2+(6n+1)^2+3p^2]^{-s}=C$$

$$\sum(-1)^{m+n}[(6m+1)^2+(6n+1)^2+6p^2]^{-s}=2^{-s}C$$

$$\sum(-1)^{m}[(6m+1)^2+12n^2+3(4p+1)^2]^{-s}=4^{-s}C$$

$$\sum(-1)^{m+[n(n+1)/2]}[3(4m+1)^2+(6n+1)^2+12p^2]^{-s}=4^{-s}C$$

$$\sum(-1)^{m}[2(6m+1)^2+3(4n+1)^2+3(4p+1)^2]^{-s}=8^{-s}C$$

$$\sum(-1)^{[m(m+1)/2]+[n(n+1)/2]}[(6m+1)^2+(6n+1)^2+6(4p+1)^2]^{-s}=8^{-s}C$$

$$\sum(-1)^{m+[n(n+1)/2]}[4(6m+1)^2+(6n+1)^2+3(4p+1)^2]^{-s}=8^{-s}C$$

$$\sum(-1)^{m+n}[12m^2+(6n+1)^2+12p^2]^{-s}=D$$

$$\sum(-1)^{m+n+p}[24m^2+24n^2+(6p+1)^2]^{-s}=D$$

$$\sum(-1)^{m+n}[(6m+1)^2+(6n+1)^2+24p^2]^{-s}=2^{-s}D$$

**Table 5.** (continued)

$$\sum(-1)^{m+n+[p(p+1)/2]}[48m^2+(6n+1)^2+(6p+1)^2]^{-s}=2^{-s}D$$
$$\sum(-1)^{m+[n(n+1)/2]+[p(p+1)/2]}[24m^2+(6n+1)^2+(6p+1)^2]^{-s}=2^{-s}D$$
$$\sum(-1)^{m}[(6m+1)^2+48n^2+3(4p+1)^2]^{-s}=4^{-s}D$$
$$\sum(-1)^{m+[n(n+1)/2]}[3(4m+1)^2+(6n+1)^2+48p^2]^{-s}=4^{-s}D$$
$$\sum(-1)^{[m(m-1)/2]+[n(n+1)/2]+[p(p+1)/2]}[3(4m+1)^2+3(4n+1)^2+2(6p+1)^2]^{-s}=8^{-s}D$$
$$\sum(-1)^{m}[(4m+1)^2+8n^2+8p^2]^{-s}=E$$
$$\sum(-1)^{m}[16m^2+8n^2+(4p+1)^2]^{-s}=E$$
$$\sum(-1)^{m+[n(n+1)/2]}[2(6m+1)^2+(6n+1)^2+24p^2]^{-s}=3^{-s}E$$
$$\sum(-1)^{[m(m+1)/2]+[n(n+1)/2]+[p(p+1)/2]}[(6m+1)^2+(6n+1)^2+(6p+1)^2]^{-s}=3^{-s}E$$
$$\sum(-1)^{m(m-1)/2}[(6m+1)^2+72n^2+8(3p+1)^2]^{-s}=9^{-s}E$$
$$\sum(-1)^{m(m-1)/2}\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)[(2m+1)^2+8n^2+8p^2]^{-s}=2^{-1}E$$
$$\sum[2(-1)^{m+n}-1](-1)^{p(p+1)/2}[24m^2+72n^2+(6p+1)^2]^{-s}=(1-3^{1-2s})E$$
$$\sum[2(-1)^{m+n}-1](-1)^{p(p+1)/2}[8m^2+24n^2+(6p+1)^2]^{-s}=(1-3^{2-2s})E$$
$$\sum(-1)^{m(m+1)/2}[(6m+1)^2+24n^2+24p^2]^{-s}=L_{-24}(2s-1)$$

## 5. Discussion

Although an attempt to systematise these results has been made, it is evident that complete success has not been achieved. The evaluation of three-dimensional sums depends essentially on finding $\Theta'$ as a single term in Eulerian form, and no systematic way of doing this is known or is necessarily available. The factors of 24 also appear to play a central role in these matters. For example, it is not possible to put $\theta(5,1)$ in Eulerian form. However, certain products of pairs and quadruplets of $\Theta$s could be expressed in this way allowing the evaluation of some unknown two- and four-dimensional sums to be evaluated. For example, it was found that

$$\theta(5,1)\theta(5,2)|q^{1/5}=q\frac{(2)^2(5)(20)}{(1)(4)}.$$

But Ramanujan in his notebooks has given (Berndt 1986)

$$\theta_3^2-\theta_3^2(q^5)=4q\frac{(2)^2(5)(20)}{(1)(4)} \tag{21}$$

and the Mellin transform of the LHS of (19) is easily found. Hence

$$\sum\sum[(5m+1)^2+(5n+2)^2]^{-s}=5^{-s}(1-5^{-s})L_1(s)L_{-4}(s).$$

Many similar results involving double and quadruple sums have been found and will be reported on later.

Clearly the processes described might be carried further. Thus one could define members of $\Theta''$ and $X''$ as, for example

$$\theta''(k, l) = \sum (kn+l)^2 q^{(kn+l)^2}$$

There would be no difficulty in finding the Mellin transforms of such quantities. Thus

$$\bar{\theta}''(6,1) = \sum (-1)^n (6n+1)^2 q^{(6n+1)^2} = q^{1^2} - 5^2 q^{5^2} - 7^2 q^{7^2} + 11^2 q^{11^2} \ldots$$

$$M\bar{\theta}''(6,1) = 1 - 5^{2s-2} - 7^{2s-2} + 11^{2s-2} \ldots = L_{12}(2s-2)$$

and the Mellin transforms of members of $\Theta''$ and $X''$ obviously yield positive parity $L$-series of order $2s-2$. However, the problem of obtaining $\Theta''$ as a single Eulerian product seems even more intractable than that for $\Theta'$, and so far no success with any $\Theta''$ has been achieved. If Eulerian or Jacobian expressions for $\Theta''$ could be obtained, then certain five-dimensional sums could be found in closed form, and the progression to higher odd dimensions is obvious.


## Acknowledgments

This research was supported by a grant from the Leverhulme Trust. The hospitality of the Physics Department of Kings College, London was most welcome. Many discussions on this and related topics with M L Glasser (Clarkson University) and J Borwein (Dalhousie) have been much appreciated.


## Appendix

Dirichlet $L$-series with real characters are defined by

$$L_d = L_d(s) = \sum_1^\infty \chi_d(k) k^{-s}$$

where $d$, an integer, will be referred to as the modulus and $s$ is the degree or order if $s$ is an integer. $\chi_d(k)$ is called a character modulo $d$ if

$$\chi_d(1) = 1 \qquad \chi_d(k) = \chi_d(k+d)$$

$$\chi_d(m)\chi_d(n) = \chi_d(mn) \qquad \text{for all } m, n$$

$$\chi_d(k) = 0 \qquad \text{if } d, k \text{ are not relatively prime.}$$

The number of independent $L$-series depends on $d$ as follows. Let $P$ be odd and square-free. Then

(i) if $d = P$ (e.g. 1, 3, 5 ...) there is just one primitive $L$-series

(ii) if $d = 4P$ (e.g. 4, 12, 20 ...) there is just one primitive $L$-series

(iii) if $d = 8P$ (e.g. 8, 24, 40 ...) there are two primitive $l$-series

(iv) if $d = 2P$, $2^\alpha P$ where $\alpha > 3$ or $P$ is not square-free (e.g. 2, 6, 9, 10 ...) there are no primitive $L$-series.

The $L$-series divide into two types according to whether $\chi_d(d-1) = \pm 1$. If $\chi_d(d-1) = 1$, the $L$-series are said to have positive parity and are denoted $L_d$. If $\chi_d(d-1) = -1$, the $L$-series are said to have negative parity and are denoted $L_{-d}$. The parity of an $L$-series is determined by $d$ as follows.

If $d=P\equiv 1 \pmod 4$ the $L$-series has positive parity.
If $d=P\equiv 3 \pmod 4$ the $L$-series has negative parity.
If $d=4P$ with $P\equiv 3 \pmod 4$ the $L$-series has positive parity.
If $d=4P$ with $P\equiv 1 \pmod 4$ the $L$-series has negative parity.
If $d=8P$ there is an $L$-series of each type.

If $s$ is a positive integer then explicit formulae for $L_{-d}(2s-1)$ and $L_d(2s)$ may be established. They are

$$L_{-d}(2s-1)=\frac{(-1)^{s-1}2^{2s-2}d^{-1/2}\pi^{2s-1}}{(2s-1)!}\sum_{n=1}^{d}\chi_d(n)B_{2s-1}\left(1-\frac{n}{d}\right)$$

$$L_d(2s)=\frac{(-1)^{s-1}2^{2s-1}d^{-1/2}\pi^{2s}}{(2s)!}\sum_{n=1}^{d}\chi_d(n)B_{2s}\left(1-\frac{n}{d}\right)$$

where the Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Since both $d$ and $n$ are positive integers, $B_m(1-n/d)$ is a rational number, hence $L_{-d}(2s-1)$ is $Rd^{1/2}\pi^{2s-1}$ and $L_d(2s)$ is $R'd^{1/2}\pi^{2s}$, where $R$ and $R'$ are rational.

The following $L$-series appear in this paper:

$L_1=1^{-s}+2^{-s}+3^{-s}+4^{-s}\dots$ (the Riemann zeta function)

$$L_{-3}=1^{-s}-2^{-s}+4^{-s}-5^{-s}\dots$$

$$L_{-4}=1^{-s}-3^{-s}+5^{-s}-7^{-s}\dots$$

$$L_{-8}=1^{-s}+3^{-s}-5^{-s}-7^{-s}\dots$$

$$L_8=1^{-s}-3^{-s}-5^{-s}+7^{-s}\dots$$

$$L_{12}=1^{-s}-5^{-s}-7^{-s}+11^{-s}\dots$$

$$L_{-24}=1^{-s}+5^{-s}+7^{-s}+11^{-s}-13^{-s}-17^{-s}-19^{-s}-23^{-s}\dots$$

$$L_{24}=1^{-s}+5^{-s}-7^{-s}-11^{-s}-13^{-s}-17^{-s}+19^{-s}+23^{-s}\dots.$$

## References

Berndt B C 1986 Private communication (Chapter 19 of Ramanujan's second notebook)
Borwein J M and Borwein P B 1986 *Pi and the AGM* (New York: Wiley)
Forrester P and Glasser M L 1982 *J. Phys. A: Math. Gen* **15** 911-4
Glasser M L 1973 *J. Math. Phys.* **14** 409-13
Glasser M L and Zucker I J 1980 *Theoretical Chemistry: Advances and Perspectives* vol 5, ed H Eyring and D Henderson (New York: Academic) pp 67-139
Gordon B 1961 *Quart. J. Math.* **12** 285-90
Hardy G H and Wright E M 1980 *An Introduction to the Theory of Numbers* (Oxford: Clarendon)
Kac V G 1978 *Adv. Math.* **30** 85-136
Lorenz L 1871 *Tidsskrift Math.* **1** 97-114
Ramanujan S 1916 *Trans. Camb. Phil. Soc.* **22** 169-84
—— 1957 *Notebooks* (Bombay: Tata Institute of Fundamental Research) (2 volumes)
Smith H J S 1865 *Report on the Theory of Numbers. Part VI* (New York: Chelsea)
Whittaker E T and Watson G N 1946 *A Course of Modern Analysis* (Cambridge: Cambridge University Press) 4th edn
Zucker I J 1975 *J. Math. Phys.* **16** 2189-91
—— 1987 *J. Phys. A: Math. Gen* **20** L13-7
Zucker I J and Robertson M M 1976 *J. Phys. A: Math. Gen* **9** 1215-25